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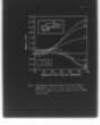
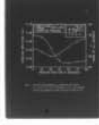
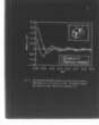
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MUTUAL COUPLING ANALYSIS IN A FINITE PLANAR RECTANGULAR WAVEGUI--ETC(U)
JUN 78 J LUZWICK, R F HARRINGTON N00014-76-C-0225

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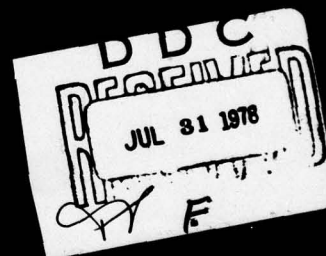
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A solution is presented for the calculation of mutual coupling between elements in a finite planar rectangular waveguide antenna array. The aperture dimensions can be less than the feeding waveguide dimensions and the elements are uniformly spaced in each of two directions in either a rectangular or isosceles triangular lattice. The problem is formulated as an integral equation by requiring the transverse electric and magnetic fields to be continuous across the apertures. The integral equation is solved by the method of moments for the equivalent magnetic current in the aperture region. A		

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single expansion function is used to approximate the electric field in each aperture. For apertures that are close together the normal quadruple half-space admittance integral is reduced to single integrals which are evaluated numerically. For apertures farther apart, a double numerical integral solution is used. In addition, the symmetry property of the symmetric Toeplitz admittance matrix for the linear array case and the symmetric block-Toeplitz admittance matrix for the rectangular array case is utilized when solving for the unknown magnetic current coefficients.

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I. INTRODUCTION

Finite array analysis is useful both for determining the characteristics of finite size arrays and also for determining edge effects for elements close to or at the edge of very large arrays. In this report the solution to the finite array problem is expressed in terms of an aperture half-space admittance matrix and coupling coefficients (scattering parameters) between elements of the array.

The problem of calculating the mutual coupling between apertures in a perfectly conducting ground plane previously has been studied by other authors [1-7]. Borgiotti [1] obtained an expression for the mutual admittance between two identical radiating apertures in the form of a Fourier transform of a function strictly related to the power radiation pattern of the element.

In the first of two papers [2-3], Mailloux found the near field coupling between two collinear open-ended waveguide slots by formulating the problem as a set of simultaneous integral equations and solving the resulting equations approximately by expanding the aperture field in a Fourier series. In the second paper, Mailloux found the near field coupling between two closely spaced open-ended square waveguide slots by a first-order analysis which is based on the method of moments using a single-mode approximation to the aperture field. He also gave an improved first-order analysis which uses a higher order mode solution.

A paper by Cha and Hsiao [4] and dissertation by Hidayet [5] treated the finite rectangular waveguide array problem, while papers

by Steyskal [6] and Bailey and Bostian [7] dealt with finite circular waveguide arrays. Steyskal's method is similar to the one used in this report. He found the scattering matrix for a finite circular waveguide array by using two dominant orthogonal modes as expansion functions in a method of moments solution.

Formulas for the computation of the aperture half-space admittance and scattering matrices of a finite planar rectangular waveguide array are derived in this report. The procedure assumes a cosine aperture electric field. For apertures that are close together the normal quadruple half-space admittance integral is analytically reduced to single integrals which are then evaluated numerically. For apertures farther apart, the quadruple half-space admittance integral is analytically reduced to double integrals which are evaluated numerically.

The general theory of solution is an extension of that presented in a previous paper [8]. The basic procedure is an application of the method of moments to an integral equation formulation of the problem. The unknowns to be determined are the equivalent magnetic current coefficients. The magnetic current is equal to the tangential electric field in the aperture regions, rotated 90° . The Topelitz property of the admittance matrices for the linear and rectangular lattice arrays is utilized when solving for the unknown magnetic current coefficients. The computer program which uses the solution derived in this report will be given in a subsequent report [15].

II. STATEMENT OF THE PROBLEM

Figure 1 shows the problem to be considered and defines the coordinates and parameters to be used. The infinitely conducting plate covers the entire $z = 0$ plane except for the apertures which are rectangular in shape with side lengths a' and b' in the x and y directions, respectively. The feeding waveguides have inside side lengths of a and b in the x and y directions, respectively. Note that all feeding waveguides have the same dimensions (a,b) and all the apertures have the same dimensions (a',b') . Also z is less than zero in the waveguide region and z is greater than zero in the half-space region. The excitations of the waveguides are sources which produce the dominant mode wave traveling toward the apertures. The waveguides are assumed terminated in matched loads when viewed from the aperture.

We first consider the problem of a single waveguide-fed aperture radiating into a half-space region (see Fig. 2a). The equivalence principle [9, Sec. 3-5] is used to divide this problem into two separate regions as follows (see Fig. 2b). The aperture is covered by an electric conductor. The fields in the waveguide region are produced by the impressed sources \underline{J}^{imp} , \underline{M}^{imp} , and the equivalent magnetic current \underline{M}

$$\underline{M} = \underline{n} \times \underline{E} \quad (1)$$

over the aperture region with the aperture covered by an electric conductor. The fields in the half-space region are produced by the equivalent magnetic current, $-\underline{M}$, with the aperture covered by an

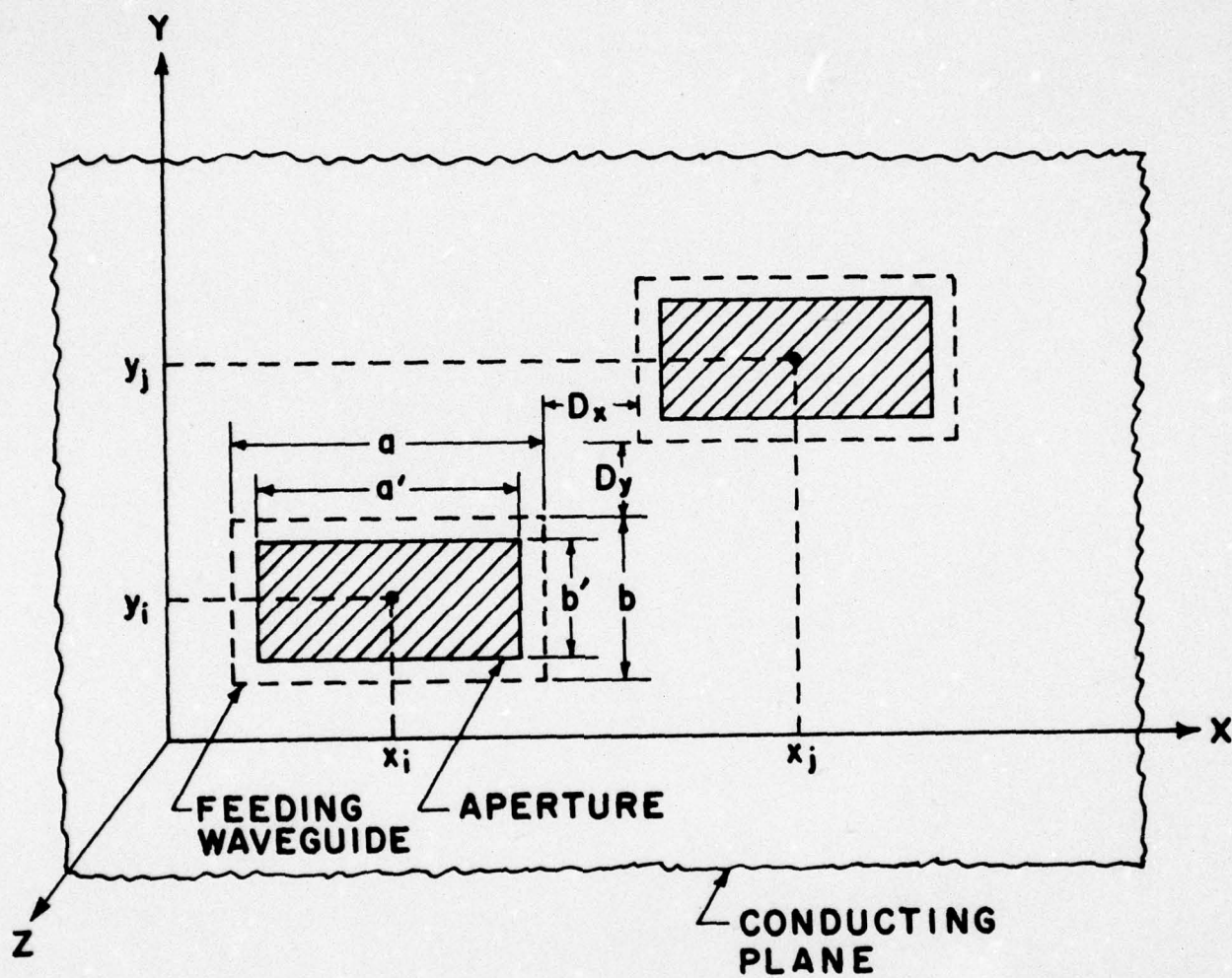
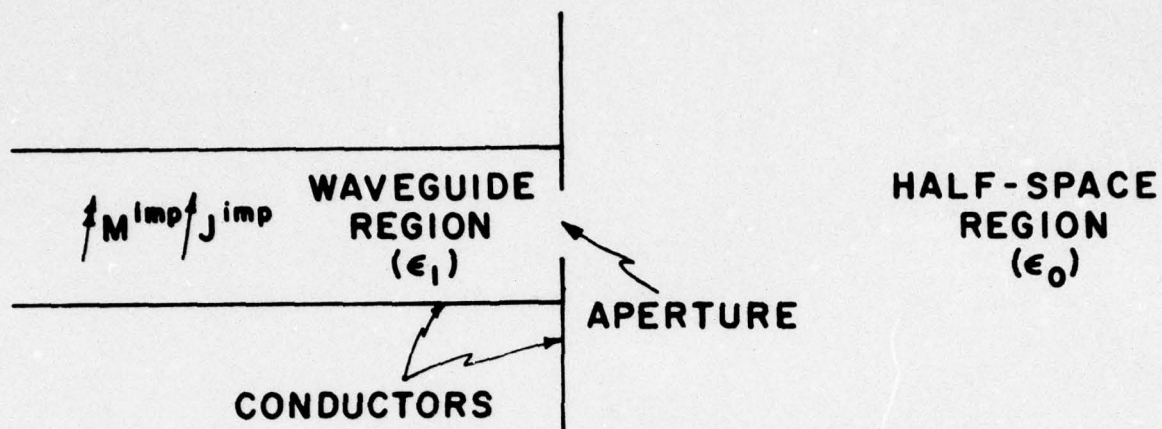
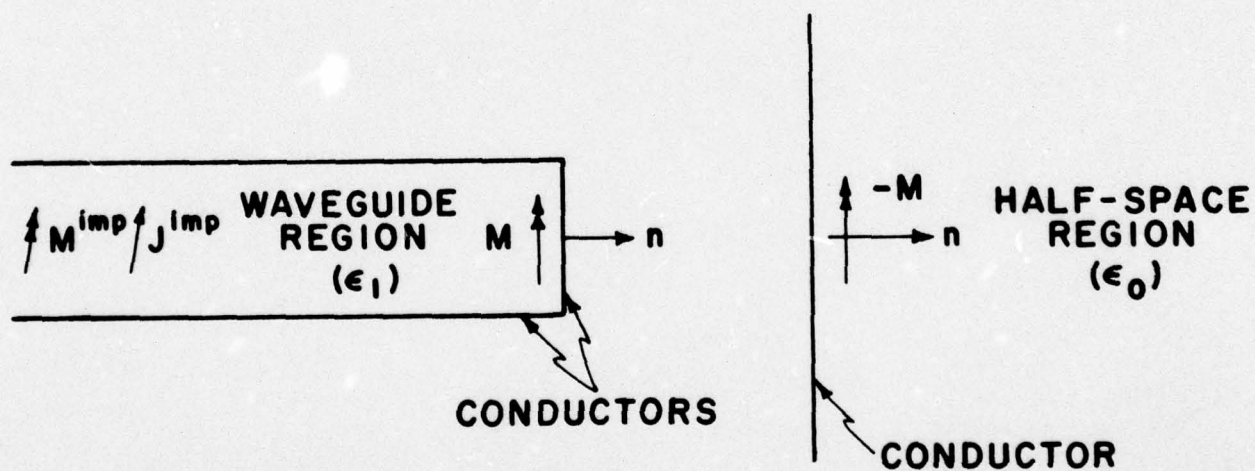


Fig. 1. Waveguide-fed rectangular apertures in a perfectly conducting plane.



(a) ORIGINAL PROBLEM



(b) EQUIVALENT PROBLEM

Fig. 2. A single waveguide-fed aperture radiating into half-space bounded by an electric conductor.

electric conductor. The condition that the equivalent magnetic current in the waveguide region is $+\underline{\underline{M}}$ and in the half-space region $-\underline{\underline{M}}$ ensures that the tangential component of electric field is continuous across the aperture.

Another necessary boundary condition is the continuity of the tangential component of magnetic field across the aperture. The tangential magnetic field over the aperture on the waveguide side, $\underline{\underline{H}}_t^{wg}$, is equal to

$$\underline{\underline{H}}_t^{wg} = \underline{\underline{H}}_t^{imp} + \underline{\underline{H}}_t^{wg}(\underline{\underline{M}}) \quad (2)$$

where

$\underline{\underline{H}}_t^{imp}$ is the tangential magnetic field due to impressed sources

$\underline{\underline{H}}_t^{wg}(\underline{\underline{M}})$ is the tangential magnetic field due to the equivalent magnetic source $\underline{\underline{M}}$.

On the half-space side of the aperture we have

$$\underline{\underline{H}}_t^{hs} = \underline{\underline{H}}_t^{hs}(-\underline{\underline{M}}) = -\underline{\underline{H}}_t^{hs}(\underline{\underline{M}}) \quad (3)$$

The last equality in (3) is a consequence of the linearity of the $\underline{\underline{H}}_t^{hs}$ operator. Note that $\underline{\underline{H}}_t^{imp}$, $\underline{\underline{H}}_t^{wg}(\underline{\underline{M}})$, and $\underline{\underline{H}}_t^{hs}(\underline{\underline{M}})$ are all computed with an electric conductor covering the aperture. The true solution is obtained when $\underline{\underline{H}}_t^{wg}$ of (2) equals $\underline{\underline{H}}_t^{hs}$ of (3) or

$$\underline{\underline{H}}_t^{wg}(\underline{\underline{M}}) + \underline{\underline{H}}_t^{hs}(\underline{\underline{M}}) = -\underline{\underline{H}}_t^{imp} \quad (4)$$

This is the basic operator equation for determining the equivalent magnetic current $\underline{\underline{M}}$.

Although (4) was derived for a single waveguide in a ground plane, it applies equally well for an N-element waveguide array. In this case the waveguide region is considered to include all of the waveguides and the half-space region is as before. For the multiple aperture case, (4) becomes

$$\underline{H}_{t_i}^{wg}(\underline{M}^i) + \sum_{j=1}^N \underline{H}_{t_i}^{hs}(\underline{M}^j) = -\underline{H}_{t_i}^{imp} \quad (5)$$

$$i = 1, 2, \dots, N.$$

In (5), \underline{M}^j is the equivalent magnetic current for the jth aperture. The subscript i denotes magnetic field evaluation in the ith aperture. Let

$$\underline{M}^j = V_j \underline{M}_j \quad (6)$$

where V_j is a complex constant to be determined and \underline{M}_j is an expansion function to be specified. Substituting (6) into (5), we obtain

$$V_i \underline{H}_{t_i}^{wg}(\underline{M}_i) + \sum_{j=1}^N V_j \underline{H}_{t_i}^{hs}(\underline{M}_j) = -\underline{H}_{t_i}^{imp} \quad (7)$$

$$i = 1, 2, \dots, N.$$

Next, define the symmetric product $\langle A, B \rangle$ of two vectors A and B by

$$\langle A, B \rangle = \iint \underline{A} \cdot \underline{B} \, ds \quad (8)$$

where the integral is over all aperture regions. Also define a set of testing functions $\{\underline{W}_i, i=1, 2, \dots, N\}$ which may or may not be equal to the expansion functions. Then, taking the symmetric product of (7) with the testing function \underline{W}_i , we obtain

$$V_i \langle W_i, H_{t_i}^{wg}(M_i) \rangle + \sum_{j=1}^N V_j \langle W_i, H_{t_i}^{hs}(M_j) \rangle = - \langle W_i, H_{t_i}^{imp} \rangle \quad (9)$$

$$i = 1, 2, \dots, N.$$

Solution of this set of linear equations determines the coefficients V_j and, therefore, the equivalent magnetic current \underline{M}_j^j .

Equation (9) can be rewritten in matrix notation as follows:

Define an admittance matrix for the waveguide regions as

$$[Y_{ij}^{wg}] = [\delta_{ij} \langle -W_i, H_{t_i}^{wg}(M_i) \rangle]_{N \times N} \quad (10)$$

where δ_{ij} is the Kronecker-delta function

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (11)$$

and for the half-space region as

$$[Y_{ij}^{hs}] = [\langle -W_i, H_{t_i}^{hs}(M_j) \rangle]_{N \times N}. \quad (12)$$

The minus signs are placed in (10) and (12) on the basis of power considerations. Define a source vector

$$\vec{I}^{imp} = [\langle W_i, H_{t_i}^{imp} \rangle]_{N \times 1} \quad (13)$$

and a coefficient vector

$$\vec{V} = [V_i]_{N \times 1}. \quad (14)$$

The resulting matrix equation which is equivalent to (9) is

$$[Y^{wg} + Y^{hs}] \vec{V} = \vec{I}^{imp}. \quad (15)$$

The physical interpretation of (15) is that of two generalized admittance networks, $[Y^{wg}]$ and $[Y^{hs}]$, in parallel with the current source \vec{I}^{imp} . By inverting (15), we obtain the resulting voltage vector \vec{V} which is the vector of coefficients which determines \underline{M}

$$\vec{V} = [Y^{wg} + Y^{hs}]^{-1} \vec{I}^{imp}. \quad (16)$$

The expansion (\underline{M}_i) and testing (\underline{W}_i) functions are defined as

$$\underline{M}_i = \underline{W}_i = -\underline{u}_x P_i(x, y) \cos \frac{\pi}{a'} (x - x_i) \quad (17)$$

where

$$P_i(x, y) = \begin{cases} \sqrt{\frac{2}{a'b'}} \begin{cases} x_i - a'/2 < x < x_i + a'/2 \\ y_i - b'/2 \leq y \leq y_i + b'/2 \end{cases} \\ 0 \quad \text{all other } x, y. \end{cases} \quad (18)$$

The equivalent magnetic current over the i th aperture region is

$$\begin{aligned} \underline{M}_i^1 &= V_i \underline{M}_i \\ &= -\underline{u}_x V_i P_i(x, y) \cos \frac{\pi}{a'} (x - x_i). \end{aligned} \quad (19)$$

The electric field in the i th aperture is

$$\underline{E}_i = \underline{u}_y V_i P_i(x, y) \cos \frac{\pi}{a'} (x - x_i). \quad (20)$$

We choose the minus sign in the magnetic current expression (19) to avoid an explicit minus sign in our expression for aperture electric field (20).

III. ADMITTANCE FORMULATION

a) Determination of Y_{ij}^{wg} -

To evaluate the aperture admittance (10) in the waveguide region, we consider a single expansion function \underline{M}_1 on the $z = 0$ plane in the i th waveguide region. The tangential field produced by \underline{M}_1 can be expressed in modal form as [9, Sec. 8-1]

$$\begin{aligned} \underline{E}_t^{wg}(\underline{M}_1) &= \sum_k A_{1k} e^{\gamma_k z} \underline{e}_k \\ \underline{H}_t^{wg}(\underline{M}_1) &= - \sum_k A_{1k} Y_k e^{\gamma_k z} \underline{u}_z \times \underline{e}_k \end{aligned} \quad (21)$$

where A_{1k} are modal amplitudes, γ_k are modal propagation constants, Y_k are modal characteristic admittances, and \underline{e}_k are normalized modal vectors. The modal vector orthogonality relationship is

$$\iint_{\text{guide}} \underline{e}_i \cdot \underline{e}_j \, ds = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad (22)$$

where the integration is over the waveguide cross section. At $z = 0$, we have

$$\underline{M}_1 = \underline{u}_z \times \underline{E}_t^{wg}(\underline{M}_1) \Big|_{z=0} = \sum_k A_{1k} \underline{u}_z \times \underline{e}_k. \quad (23)$$

Multiply each side of this equation scalarly by $\underline{u}_z \times \underline{e}_j$ and integrate over the waveguide cross section obtaining

$$\iint_{\text{guide}} \underline{M}_1 \cdot \underline{u}_z \times \underline{e}_j \, ds = \sum_k A_{1k} \iint_{\text{guide}} (\underline{u}_z \times \underline{e}_k) \cdot (\underline{u}_z \times \underline{e}_j) \, ds. \quad (24)$$

By orthogonality (22), all terms of the summation are zero except the $j=k$ term. Hence

$$A_{ik} = \iint_{\text{apert.}} \underline{\underline{M}}_i \cdot \underline{\underline{u}}_z \times \underline{\underline{e}}_k \, ds. \quad (25)$$

We have replaced the integral over the waveguide cross section by one over the aperture since $\underline{\underline{M}}_i$ exists only in the aperture region. Substituting the second equation of (21) evaluated at $z = 0$ into (10), we obtain the diagonal elements of the waveguide admittance matrix, $[Y^{wg}]$,

$$Y_{ij}^{wg} = \delta_{ij} \sum_k A_{ik} Y_k \iint_{\text{apert.}} \underline{\underline{W}}_i \cdot \underline{\underline{u}}_z \times \underline{\underline{e}}_k \, ds. \quad (26)$$

Since $\underline{\underline{M}}_i = \underline{\underline{W}}_i$, (26) becomes

$$Y_{ij}^{wg} = \delta_{ij} \sum_k A_{ik}^2 Y_k. \quad (27)$$

The set $\underline{\underline{e}}_k$ of modes for the rectangular waveguide is split into a set $\underline{\underline{e}}_k^{TE}$ of TE modes given by [9, Equations (8-34), (3-86), and (3-89) and Section 4-3]

$$\begin{aligned} \underline{\underline{e}}_{m+n(L_m+1)}^{TE} &= \sqrt{\frac{ab \epsilon_m \epsilon_n}{(mb)^2 + (na)^2}} \left[\underline{\underline{u}}_x \frac{n}{b} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\ &\quad \left. - \underline{\underline{u}}_y \frac{m}{a} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right] \\ m &= 0, 1, 2, \dots, L_m \\ n &= 0, 1, 2, \dots, L_n \quad m+n \neq 0 \\ \epsilon_m &= \begin{cases} 1 & m = 0 \\ 2 & m = 1, 2, \dots \end{cases} \end{aligned} \quad (28)$$

and a set $\underline{\underline{e}}_k^{TM}$ of TM modes given by

$$e_{m+(n-1)L_m}^{TM} = 2 \sqrt{\frac{ab}{(mb)^2 + (na)^2}} \left[u_x \frac{m}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + u_y \frac{n}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right] \quad (29)$$

$$m = 1, 2, 3, \dots, L_m$$

$$n = 1, 2, 3, \dots, L_n$$

Note that equations (28) and (29) are valid only when the origin is at a corner of the waveguide.

Substituting (17), (28), and (29) into (25), we obtain

$$A_{ik}^{TE} = \iint_{\text{apert.}} \underline{M}_i \cdot \underline{u}_z \times \underline{e}_k^{TE} ds$$

$$= - \frac{m\pi}{k_1 a} \sqrt{\frac{2\epsilon_m \epsilon_n}{aa' bb'}} I(m, n) \quad (30)$$

$$k = m+n(L_m+1) \begin{cases} m=0, 1, 2, \dots, L_m \\ n=0, 1, 2, \dots, L_n \end{cases} \quad m+n \neq 0$$

and

$$A_{ik}^{TM} = \iint_{\text{apert.}} \underline{M}_i \cdot \underline{u}_z \times \underline{e}_k^{TM} ds$$

$$= \frac{2n\pi}{k_1 b} \sqrt{\frac{2}{aa' bb'}} I(m, n) \quad (31)$$

$$k = m+(n-1)L_m \begin{cases} m=1, 2, 3, \dots, L_m \\ n=1, 2, 3, \dots, L_n \end{cases}$$

where

$$k_1 = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad (32)$$

$$\begin{aligned}
 I(m,n) &= \int_{(b-b')/2}^{(b+b')/2} dy \cos \frac{n\pi y}{b} \int_{(a-a')/2}^{(a+a')/2} dx \cos \frac{\pi}{a'} (x-a/2) \sin \frac{m\pi x}{a} \\
 &= -\frac{2b'}{\pi a'} \frac{1}{\left(\frac{m^2}{2} - \frac{1}{a'^2}\right)} \sin \frac{m\pi}{2} \cos \frac{n\pi}{2} \cos \frac{m\pi a'}{2a} \frac{\sin \frac{n\pi b'}{2b}}{\frac{n\pi b'}{2b}}. \quad (33)
 \end{aligned}$$

Note that the introduction of e_k^{TE} and e_k^{TM} has split the coefficients (25) into A_{ik}^{TE} and A_{ik}^{TM} . If $(a/a' = m)$ in (33), $(\cos \frac{m\pi a'}{2a} / (\frac{m^2}{2} - \frac{1}{a'^2}))$ is to be replaced by its limit $(-\pi a'^2/4)$. If n is zero in (33), $(\sin(0)/0)$ is to be replaced by unity.

The characteristic admittances Y_i of a rectangular waveguide with relative dielectric constant ϵ_r and relative permeability unity are classified as either TE admittances Y_i^{TE} or TM admittances Y_i^{TM} given by [9, Section 4-3],

$$Y_i^{TE} = \begin{cases} -\frac{1}{\eta} \sqrt{\left(\frac{k_i}{k}\right)^2 - \epsilon_r} & k\sqrt{\epsilon_r} < k_i \\ \frac{1}{\eta} \sqrt{\epsilon_r - \left(\frac{k_i}{k}\right)^2} & k\sqrt{\epsilon_r} > k_i \end{cases} \quad (34)$$

$$i = m+n(L_m+1) \begin{cases} m=0,1,2,\dots,L_m \\ n=0,1,2,\dots,L_n \end{cases} \quad m+n \neq 0$$

and

$$Y_i^{TM} = \begin{cases} \frac{1}{\eta} \frac{\epsilon_r}{\sqrt{\left(\frac{k_i}{k}\right)^2 - \epsilon_r}} & k\sqrt{\epsilon_r} < k_i \\ \frac{1}{\eta} \frac{\epsilon_r}{\sqrt{\epsilon_r - \left(\frac{k_i}{k}\right)^2}} & k\sqrt{\epsilon_r} > k_i \end{cases} \quad (35)$$

$$i = m+(n-1)L_m \begin{cases} m=1,2,3,\dots,L_m \\ n=1,2,3,\dots,L_n \end{cases}$$

In (34) and (35), η is the characteristic impedance of free space, k is the free space wave number, and k_1 is the cut-off wave number given by (32).

b) Determination of Y_{ij}^{hs} -

Since the apertures are covered by conductors, the $z = 0$ plane is a complete conducting plane and image theory applies. The magnetic current expansion functions are on the surface of the $z = 0$ plane. Their images are equal to them and are also on the $z = 0$ plane. The result is that $[Y^{hs}]$ is the admittance matrix obtained using expansion functions $2\vec{M}_j$ radiating into free space everywhere. Therefore (12) can be written as

$$Y_{ij}^{hs} = -2 \langle \vec{W}_i, \vec{H}_{t_i}^{fs}(\vec{M}_j) \rangle \quad (36)$$

where $\vec{H}_{t_i}^{fs}(\vec{M}_j)$ is the magnetic field in the i th aperture region produced by \vec{M}_j radiating into free space. The magnetic field $\vec{H}_{t_i}^{fs}(\vec{M}_j)$ can be expressed in terms of an electric vector potential \vec{F}_{ij} and a magnetic scalar potential ϕ_{ij} as [10]

$$\vec{H}_{t_i}^{fs}(\vec{M}_j) = -j\omega \vec{F}_{ij} - \nabla \phi_{ij} \quad (37)$$

where

$$\vec{F}_{ij} = \frac{\epsilon}{4\pi} \iint_{\text{apert.}} \vec{M}_j \frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} ds \quad (38)$$

$$\phi_{ij} = \frac{1}{4\pi\mu} \iint_{\text{apert.}} \rho_j \frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} ds \quad (39)$$

$$\rho_j = \frac{\nabla \cdot \vec{M}_j}{-j\omega} \quad (40)$$

where \underline{r} and \underline{r}' are respectively the vectors to the field and source points, ω is the angular frequency, ϵ is the permittivity of free space, μ is the permeability of free space, and k is the free space propagation constant. Substituting (37) into (36) and using (8), we obtain

$$Y_{ij}^{hs} = 2 \iint_{\text{apert.}} \underline{W}_1 \cdot (j\omega \underline{F}_{ij} + \nabla \phi_{ij}) ds. \quad (41)$$

Because of the identity

$$0 = \iint_{\text{apert.}} \nabla \cdot (\phi_{ij} \underline{W}_1) ds = \iint_{\text{apert.}} \underline{W}_1 \cdot \nabla \phi_{ij} ds + \iint_{\text{apert.}} \phi_{ij} \nabla \cdot \underline{W}_1 ds \quad (42)$$

(41) becomes

$$Y_{ij}^{hs} = 2 j\omega \iint_{\text{apert.}} (\underline{F}_{ij} \cdot \underline{W}_1 + \phi_{ij} \rho_1) ds \quad (43)$$

where

$$\rho_1 = \frac{\nabla \cdot \underline{W}_1}{-j\omega}. \quad (44)$$

Next define Y_{ij}^1 by

$$Y_{ij}^1 = 2 j\omega \iint_{\text{apert.}} \underline{F}_{ij} \cdot \underline{W}_1 ds. \quad (45)$$

Substituting (17) into (45), we obtain

$$Y_{ij}^1 = \frac{j\omega\epsilon}{\pi a' b'} \int_{y_1 - b'/2}^{y_1 + b'/2} dy \int_{x_1 - a'/2}^{x_1 + a'/2} dx \cos \frac{\pi}{a'}(x - x_1) \int_{y_j - b'/2}^{y_j + b'/2} dy' \int_{x_j - a'/2}^{x_j + a'/2} dx' \cos \frac{\pi}{a'}(x' - x_j) G(x' - x, y' - y) \quad (46)$$

where

$$G(x' - x, y' - y) = \frac{e^{-jk\sqrt{(x'-x)^2 + (y'-y)^2}}}{\sqrt{(x'-x)^2 + (y'-y)^2}}. \quad (47)$$

Consider

$$I_{ij}^y(x'-x) = \int_{y_1-b'/2}^{y_1+b'/2} dy \int_{y_j-b'/2}^{y_j+b'/2} dy' G(x'-x, y'-y). \quad (48)$$

Substituting the transformation $y' = v+y$ into (48), we obtain

$$I_{ij}^y(x'-x) = \int_{y_1-b'/2}^{y_1+b'/2} dy \int_{y_j-b'/2-y}^{y_j+b'/2-y} dv G(x'-x, v). \quad (49)$$

Interchanging the order of integration, we obtain (see Fig. 3)

$$\begin{aligned} I_{ij}^y(x'-x) &= \int_{y_j-y_1-b'}^{y_j-y_1} dv G(x'-x, v) \int_{y_1-b'/2-v}^{y_1+b'/2} dy \\ &+ \int_{y_j-y_1}^{y_j-y_1+b'} dv G(x'-x, v) \int_{y_1-b'/2}^{y_j+b'/2-v} dy. \end{aligned} \quad (50)$$

Consider

$$I_{ij}^{lx} = \int_{x_1-a'/2}^{x_1+a'/2} dx \cos \frac{\pi}{a'} (x-x_1) \int_{x_j-a'/2}^{x_j+a'/2} dx' \cos \frac{\pi}{a'} (x'-x_j) I_{ij}^y(x'-x). \quad (51)$$

Substituting the transformation $x' = u+x$ into (51), we obtain

$$I_{ij}^{lx} = \int_{x_1-a'/2}^{x_1+a'/2} dx \cos \frac{\pi}{a'} (x-x_1) \int_{x_j-a'/2-x}^{x_j+a'/2-x} du \cos \frac{\pi}{a'} (u+x-x_j) I_{ij}^y(u). \quad (52)$$

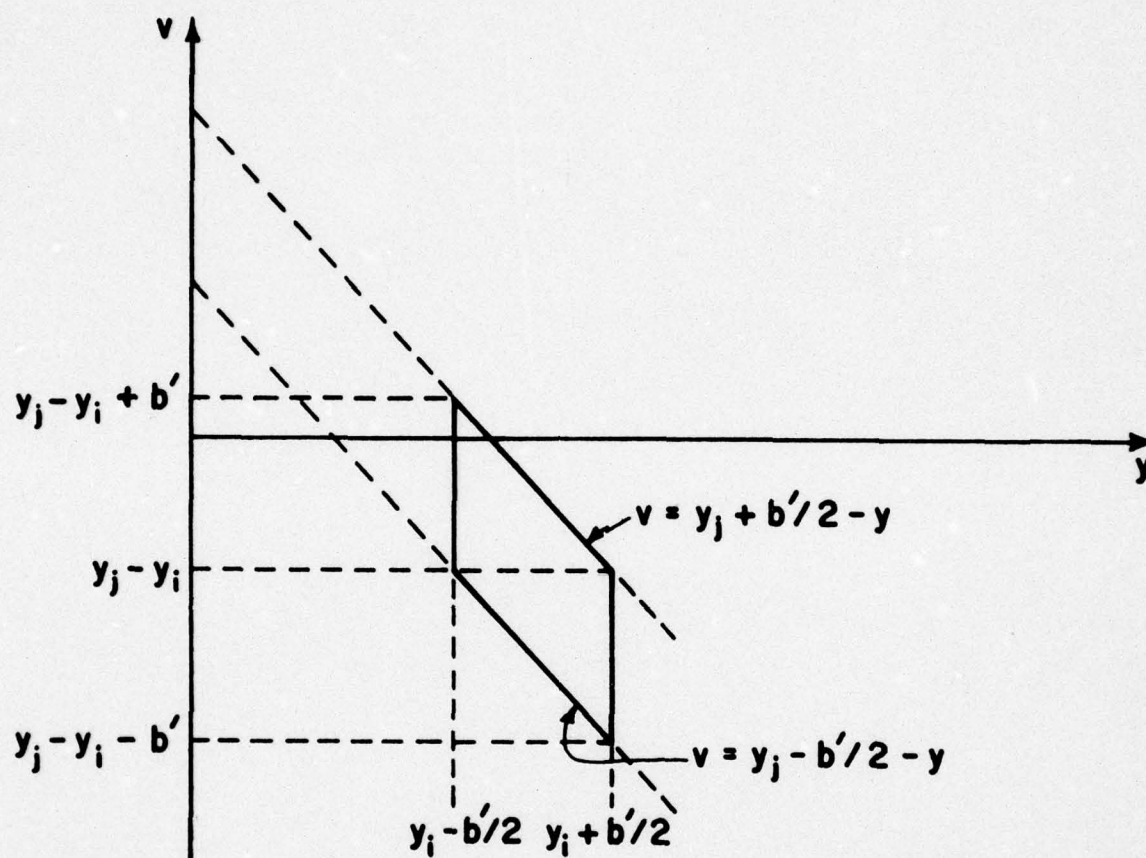


Fig. 3. Coordinate transformation.

Interchanging the order of integration, we obtain

$$\begin{aligned}
 I_{ij}^{lx} = & \int_{x_j - x_i - a'}^{x_j - x_i} du \int_{x_i - a'/2 - u}^{x_i + a'/2} dx \cos \frac{\pi}{a'} (x - x_i) \cos \frac{\pi}{a'} (u + x - x_j) \\
 & + \int_{x_j - x_i}^{x_j - x_i + a'} du \int_{x_i - a'/2}^{x_j + a'/2 - u} dx \cos \frac{\pi}{a'} (x - x_i) \cos \frac{\pi}{a'} (u + x - x_j). \quad (53)
 \end{aligned}$$

Substituting (50) and (53) into (46), we obtain

$$Y_{ij}^1 = \frac{j\omega\epsilon}{\pi a' b'} I_{ij}^{lx}. \quad (54)$$

Next define Y_{ij}^2 by

$$Y_{ij}^2 = 2j\omega \iint_{\text{apert.}} \phi_{ij} \rho_i ds. \quad (55)$$

Substituting (17) into (55), we obtain

$$Y_{ij}^2 = \frac{-j\pi}{\omega\mu a' b'} \int_{y_i - b'/2}^{y_i + b'/2} dy \int_{x_i - a'/2}^{x_i + a'/2} dx \sin \frac{\pi}{a'} (x - x_i) \int_{y_j - b'/2}^{y_j + b'/2} dy' \int_{x_j - a'/2}^{x_j + a'/2} dx' \sin \frac{\pi}{a'} (x' - x_j) G(x' - x, y' - y). \quad (56)$$

Substituting the transformations $x' = u + x$ and $y' = v + y$ into (56) and interchanging the order of integration, we obtain

$$Y_{ij}^2 = \frac{-j\pi}{\omega\mu a' b'} I_{ij}^{2x} \quad (57)$$

where

$$\begin{aligned}
I_{ij}^{2x} = & \int_{x_j - x_i - a'}^{x_j - x_i} du \int_{x_j - a'/2 - u}^{x_i + a'/2} I_{ij}^y(u) dx \sin \frac{\pi}{a'} (x - x_i) \sin \frac{\pi}{a'} (u + x - x_j) \\
& + \int_{x_j - x_i}^{x_j - x_i + a'} du \int_{x_i - a'/2}^{x_j + a'/2 - u} I_{ij}^y(u) dx \sin \frac{\pi}{a'} (x - x_i) \sin \frac{\pi}{a'} (u + x - x_j). \quad (58)
\end{aligned}$$

After a tedious but straightforward evaluation of two integrations for each of the quadruple integrals I_{ij}^{1x} and I_{ij}^{2x} , we obtain

$$\begin{aligned}
Y_{ij}^{hs} = & Y_{ij}^1 + Y_{ij}^2 \\
= & \frac{jk}{2\pi\eta a' b'} \left\{ \int_{y_j - y_i - b'}^{y_j - y_i} dv (K_1 + v) \left[\int_{x_j - x_i - a'}^{x_j - x_i} du G(u, v) [(K_2 + K_3 u) \cos \frac{\pi u}{a'} + (K_4 + K_5 u) \sin \frac{\pi u}{a'}] \right. \right. \\
& + \int_{x_j - x_i}^{x_j - x_i + a'} du G(u, v) [(K_7 - K_3 u) \cos \frac{\pi u}{a'} + (K_8 - K_5 u) \sin \frac{\pi u}{a'}] \\
& + \int_{y_j - y_i}^{y_j - y_i + b'} dv (K_6 - v) \left[\int_{x_j - x_i - a'}^{x_j - x_i} du G(u, v) [(K_2 + K_3 u) \cos \frac{\pi u}{a'} + (K_4 + K_5 u) \sin \frac{\pi u}{a'}] \right. \\
& \left. \left. + \int_{x_j - x_i}^{x_j - x_i + a'} du G(u, v) [(K_7 - K_3 u) \cos \frac{\pi u}{a'} + (K_8 - K_5 u) \sin \frac{\pi u}{a'}] \right] \right\} \quad (59)
\end{aligned}$$

where

$$K_1 = y_i - y_j + b' \quad (60)$$

$$K_2 = (1 - \frac{\pi^2}{k^2 a'^2})(x_i - x_j + a') \cos [\frac{\pi}{a'} (x_j - x_i)] + \frac{a'}{\pi} (1 + \frac{\pi^2}{k^2 a'^2}) \sin [\frac{\pi}{a'} (x_j - x_i)] \quad (61)$$

$$K_3 = (1 - \frac{\pi^2}{k^2 a'^2}) \cos [\frac{\pi}{a'} (x_j - x_i)] \quad (62)$$

$$K_4 = (1 - \frac{\pi^2}{k^2 a'^2})(x_i - x_j + a') \sin [\frac{\pi}{a'} (x_j - x_i)] - \frac{a'}{\pi} (1 + \frac{\pi^2}{k^2 a'^2}) \cos [\frac{\pi}{a'} (x_j - x_i)] \quad (63)$$

$$K_5 = (1 - \frac{\pi^2}{k^2 a'^2}) \sin [\frac{\pi}{a'} (x_j - x_i)] \quad (64)$$

$$K_6 = y_j - y_i + b' \quad (65)$$

$$K_7 = (1 - \frac{\pi^2}{k^2 a'^2})(x_j - x_i + a') \cos [\frac{\pi}{a'} (x_j - x_i)] - \frac{a'}{\pi} (1 + \frac{\pi^2}{k^2 a'^2}) \sin [\frac{\pi}{a'} (x_j - x_i)] \quad (66)$$

$$K_8 = (1 - \frac{\pi^2}{k^2 a'^2})(x_j - x_i + a') \sin [\frac{\pi}{a'} (x_j - x_i)] + \frac{a'}{\pi} (1 + \frac{\pi^2}{k^2 a'^2}) \cos [\frac{\pi}{a'} (x_j - x_i)]. \quad (67)$$

Substituting the coordinate transformations $u = \rho \cos \theta$ and $v = \rho \sin \theta$ into (59) and integrating out the ρ variable, we obtain (see Fig. 4)

$$\begin{aligned} Y_{ij}^{hs} = \frac{jk}{2\pi\eta a' b'} \{ & \int_{\theta_1}^{\theta_2} d\theta [K_1 [K_2 L_1(\theta) + K_3 L_3(\theta) + K_4 L_2(\theta) + K_5 L_4(\theta)] \\ & + [K_2 L_5(\theta) + K_3 L_7(\theta) + K_4 L_6(\theta) + K_5 L_8(\theta)]] \\ & + \int_{\theta_3}^{\theta_4} d\theta [K_1 [K_7 L_1(\theta) - K_3 L_3(\theta) + K_8 L_2(\theta) - K_5 L_4(\theta)] \\ & + [K_7 L_5(\theta) - K_3 L_7(\theta) + K_8 L_6(\theta) - K_5 L_8(\theta)]] \end{aligned} \quad (I)$$

(II)

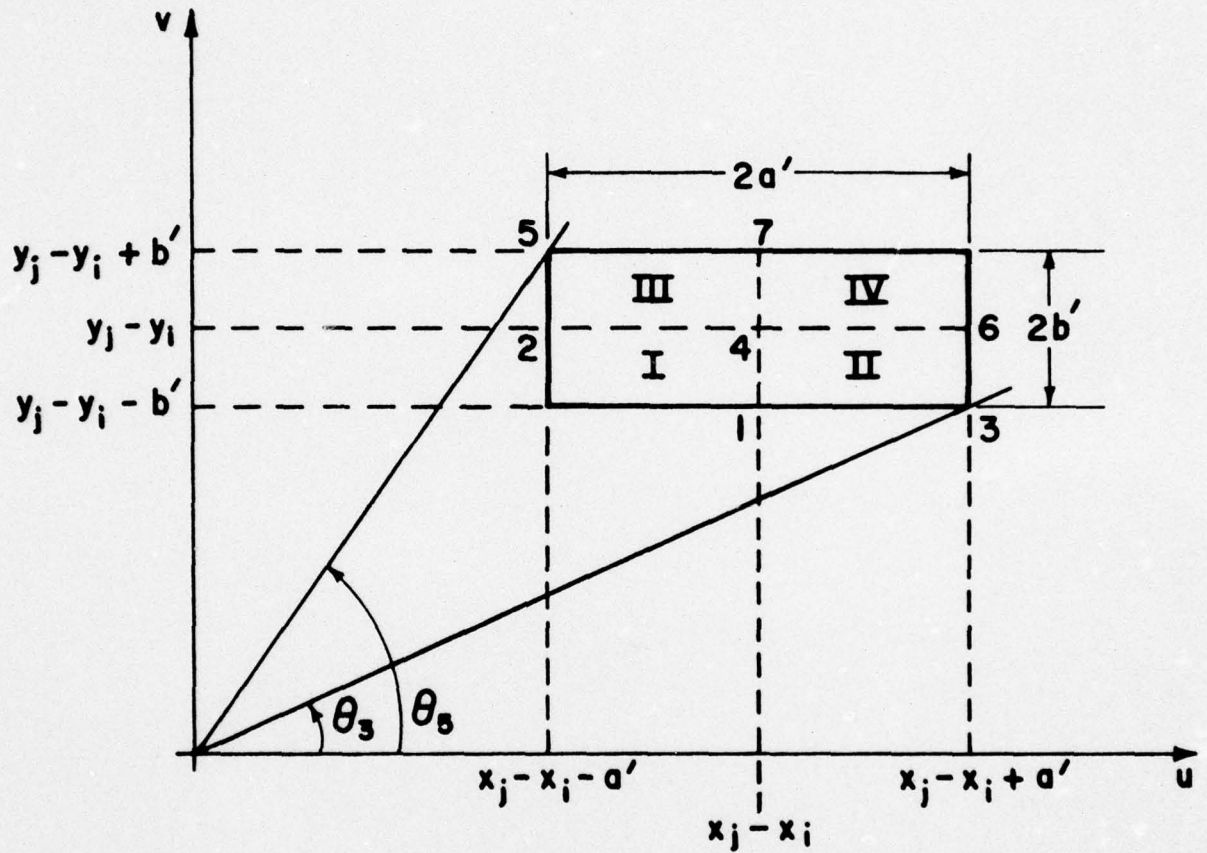


Fig. 4. Rectangular - polar coordinate integration area.

$$\begin{aligned}
& + \int_{\theta_4}^{\theta_5} d\theta [K_6 [K_2 L_1(\theta) + K_3 L_3(\theta) + K_4 L_2(\theta) + K_5 L_4(\theta)] \\
& \quad - [K_2 L_5(\theta) + K_3 L_7(\theta) + K_4 L_6(\theta) + K_5 L_8(\theta)]] \\
& \quad (III) \\
& + \int_{\theta_6}^{\theta_7} d\theta [K_6 [K_7 L_1(\theta) - K_3 L_3(\theta) + K_8 L_2(\theta) - K_5 L_4(\theta)] \\
& \quad - [K_7 L_5(\theta) - K_3 L_7(\theta) + K_8 L_6(\theta) - K_5 L_8(\theta)]]] \\
& \quad (IV)
\end{aligned} \tag{68}$$

where

$$L_1(\theta) = \frac{-j}{2} \left\{ \frac{M_1(\theta) - M_2(\theta)}{M_5(\theta)} - \frac{M_3(\theta) - M_4(\theta)}{M_6(\theta)} \right\} \tag{69}$$

$$L_2(\theta) = -\frac{1}{2} \left\{ \frac{M_1(\theta) - M_2(\theta)}{M_5(\theta)} + \frac{M_3(\theta) - M_4(\theta)}{M_6(\theta)} \right\} \tag{70}$$

$$\begin{aligned}
L_3(\theta) = \cos \theta \left\{ \frac{-j}{2} \left[\frac{\rho_2(\theta) M_1(\theta) - \rho_1(\theta) M_2(\theta)}{M_5(\theta)} - \frac{\rho_2(\theta) M_3(\theta) - \rho_1(\theta) M_4(\theta)}{M_6(\theta)} \right] \right. \\
\left. + \frac{1}{2} \left[\frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^2} + \frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^2} \right] \right\}
\end{aligned} \tag{71}$$

$$\begin{aligned}
L_4(\theta) = \cos \theta \left\{ -\frac{1}{2} \left[\frac{\rho_2(\theta) M_1(\theta) - \rho_1(\theta) M_2(\theta)}{M_5(\theta)} + \frac{\rho_2(\theta) M_3(\theta) - \rho_1(\theta) M_4(\theta)}{M_6(\theta)} \right] \right. \\
\left. - \frac{j}{2} \left[\frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^2} - \frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^2} \right] \right\}
\end{aligned} \tag{72}$$

$$L_5(\theta) = L_3(\theta) \tan \theta \tag{73}$$

$$L_6(\theta) = L_4(\theta) \tan \theta \tag{74}$$

$$\begin{aligned}
L_7(\theta) = \sin \theta \cos \theta \{ & -\frac{1}{2} \left[\frac{\rho_2^2(\theta)M_1(\theta) - \rho_1^2(\theta)M_2(\theta)}{M_5(\theta)} - \frac{\rho_2^2(\theta)M_3(\theta) - \rho_1^2(\theta)M_4(\theta)}{M_6(\theta)} \right] \\
& + \frac{\rho_2(\theta)M_1(\theta) - \rho_1(\theta)M_2(\theta)}{(M_5(\theta))^2} + \frac{\rho_2(\theta)M_3(\theta) - \rho_1(\theta)M_4(\theta)}{(M_6(\theta))^2} \\
& + j \left[\frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^3} - \frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^3} \right] \} \quad (75)
\end{aligned}$$

$$\begin{aligned}
L_8(\theta) = \sin \theta \cos \theta \{ & -\frac{1}{2} \left[\frac{\rho_2^2(\theta)M_1(\theta) - \rho_1^2(\theta)M_2(\theta)}{M_5(\theta)} + \frac{\rho_2^2(\theta)M_3(\theta) - \rho_1^2(\theta)M_4(\theta)}{M_6(\theta)} \right] \\
& - j \left[\frac{\rho_2(\theta)M_1(\theta) - \rho_1(\theta)M_2(\theta)}{(M_5(\theta))^2} - \frac{\rho_2(\theta)M_3(\theta) - \rho_1(\theta)M_4(\theta)}{(M_6(\theta))^2} \right] \\
& + \frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^3} + \frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^3} \} \quad (76)
\end{aligned}$$

$$M_1(\theta) = e^{j\rho_2(\theta)(\frac{\pi}{a'} \cos \theta - k)} \quad (77)$$

$$M_2(\theta) = e^{j\rho_1(\theta)(\frac{\pi}{a'} \cos \theta - k)} \quad (78)$$

$$M_3(\theta) = e^{-j\rho_2(\theta)(\frac{\pi}{a'} \cos \theta + k)} \quad (79)$$

$$M_4(\theta) = e^{-j\rho_1(\theta)(\frac{\pi}{a'} \cos \theta + k)} \quad (80)$$

$$M_5(\theta) = \frac{\pi}{a'} \cos \theta - k \quad (81)$$

$$M_6(\theta) = \frac{\pi}{a'} \cos \theta + k \quad (82)$$

The variables $\rho_1(\theta)$ and $\rho_2(\theta)$ are respectively the lower and upper limits of integration for the ρ integration in (59) after the transformations $u = \rho \cos \theta$ and $v = \rho \sin \theta$ are used. They are dependent on the θ variable and the integration subarea. For instance, when integrating over region IV between θ_4 and θ_7 (see Fig. 4), $\rho_1(\theta) = (x_j - x_i)/\cos \theta$ and $\rho_2(\theta) = (y_j - y_i + b')/\sin \theta$.

IV. SINGULAR POINTS OF Y^{hs}

The integrand of (68) has removable singularities when $ka' \leq \pi$. When either $|\cos \theta - 2a'| < \epsilon$ (small value) or $|\cos \theta + 2a'| < \epsilon$, we replace equations (69)-(76) by their respective limits. For an example, consider equation (69),

$$\begin{aligned}
 L_1(\theta) &= -\frac{j}{2} \left\{ \frac{M_1(\theta) - M_2(\theta)}{M_5(\theta)} - \frac{M_3(\theta) - M_4(\theta)}{M_6(\theta)} \right\} \\
 &= -\frac{j}{2} \left\{ \frac{e^{j\rho_2(\theta)(\frac{\pi}{a'} \cos \theta - k)} - e^{j\rho_1(\theta)(\frac{\pi}{a'} \cos \theta - k)}}{(\frac{\pi}{a'} \cos \theta - k)} \right. \\
 &\quad \left. - \frac{e^{-j\rho_2(\theta)(\frac{\pi}{a'} \cos \theta + k)} - e^{-j\rho_1(\theta)(\frac{\pi}{a'} \cos \theta + k)}}{(\frac{\pi}{a'} \cos \theta + k)} \right\}. \quad (69)
 \end{aligned}$$

If $|\frac{\pi}{a'} \cos \theta - k| < \epsilon$, then by retaining the first three terms of the Taylor series for the exponentials in the first half of (69), we obtain

$$\begin{aligned}
 L_1(\theta) &= -\frac{j}{2} \left\{ 1 + j\rho_2(\theta)(\frac{\pi}{a'} \cos \theta - k) - \frac{1}{2}[\rho_2(\theta)(\frac{\pi}{a'} \cos \theta - k)]^2 \right. \\
 &\quad \left. - 1 - j\rho_1(\theta)(\frac{\pi}{a'} \cos \theta - k) + \frac{1}{2}[\rho_1(\theta)(\frac{\pi}{a'} \cos \theta - k)]^2 \right. \\
 &\quad \left. - \frac{M_3(\theta) - M_4(\theta)}{M_6(\theta)} \right\}. \quad (83)
 \end{aligned}$$

If we take the limit as $\frac{\pi}{a} \cos \theta$ approaches k for the first half of (83), we obtain

$$L_1(\theta) = \frac{1}{2} (\rho_2(\theta) - \rho_1(\theta)) + \frac{j}{2} \left(\frac{M_3(\theta) - M_4(\theta)}{M_6(\theta)} \right). \quad (84)$$

If $|\frac{\pi}{a} \cos \theta + k| < \epsilon$, then by retaining the first three terms of the Taylor series for the exponentials in the second half of (69), we obtain

$$L_1(\theta) = -\frac{j}{2} \left(\frac{M_1(\theta) - M_2(\theta)}{M_5(\theta)} \right) + \frac{1}{2} (\rho_2(\theta) - \rho_1(\theta)). \quad (85)$$

If we apply the preceding procedure to equations (70)-(76), we obtain

$$L_2(\theta) = -\frac{j}{2} (\rho_2(\theta) - \rho_1(\theta)) - \frac{1}{2} \left(\frac{M_3(\theta) - M_4(\theta)}{M_6(\theta)} \right) \quad (86)$$

$$L_3(\theta) = \cos \theta \left\{ \frac{1}{4} (\rho_2^2(\theta) - \rho_1^2(\theta)) + \frac{j}{2} \left(\frac{\rho_2(\theta)M_3(\theta) - \rho_1(\theta)M_4(\theta)}{M_6(\theta)} \right) + \frac{1}{2} \left(\frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^2} \right) \right\} \quad (87)$$

$$L_4(\theta) = \cos \theta \left\{ -\frac{j}{4} (\rho_2^2(\theta) - \rho_1^2(\theta)) - \frac{1}{2} \left(\frac{\rho_2(\theta)M_3(\theta) - \rho_1(\theta)M_4(\theta)}{M_6(\theta)} \right) + \frac{j}{2} \left(\frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^2} \right) \right\} \quad (88)$$

$$L_5(\theta) = L_3(\theta) \tan \theta \quad (89)$$

$$L_6(\theta) = L_4(\theta) \tan \theta \quad (90)$$

$$L_7(\theta) = \sin \theta \cos \theta \left\{ \frac{1}{6} (\rho_2^3(\theta) - \rho_1^3(\theta)) + \frac{j}{2} \left(\frac{\rho_2^2(\theta)M_3(\theta) - \rho_1^2(\theta)M_4(\theta)}{M_6(\theta)} \right) + \frac{\rho_2(\theta)M_3(\theta) - \rho_1(\theta)M_4(\theta)}{(M_6(\theta))^2} - j \left(\frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^3} \right) \right\} \quad (91)$$

$$\begin{aligned}
L_8(\theta) = \sin \theta \cos \theta \left\{ -\frac{1}{6}(\rho_2^3(\theta) - \rho_1^3(\theta)) - \frac{1}{2} \left(\frac{\rho_2^2(\theta)M_3(\theta) - \rho_1^2(\theta)M_4(\theta)}{M_6(\theta)} \right) \right. \\
\left. + j \left(\frac{\rho_2(\theta)M_3(\theta) - \rho_1(\theta)M_4(\theta)}{(M_6(\theta))^2} \right) + \frac{M_3(\theta) - M_4(\theta)}{(M_6(\theta))^3} \right\} \quad (92)
\end{aligned}$$

for $|\frac{\pi}{a} \cos \theta - k| < \epsilon$, and

$$L_2(\theta) = -\frac{1}{2} \left(\frac{M_1(\theta) - M_2(\theta)}{M_5(\theta)} \right) + \frac{1}{2} (\rho_2(\theta) - \rho_1(\theta)) \quad (93)$$

$$L_3(\theta) = \cos \theta \left\{ -\frac{1}{2} \left(\frac{\rho_2(\theta)M_1(\theta) - \rho_1(\theta)M_2(\theta)}{M_5(\theta)} \right) + \frac{1}{2} \left(\frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^2} \right) + \frac{1}{4}(\rho_2^2(\theta) - \rho_1^2(\theta)) \right\} \quad (94)$$

$$L_4(\theta) = \cos \theta \left\{ -\frac{1}{2} \left(\frac{\rho_2(\theta)M_1(\theta) - \rho_1(\theta)M_2(\theta)}{M_5(\theta)} \right) - \frac{1}{2} \left(\frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^2} \right) + \frac{1}{4}(\rho_2^2(\theta) - \rho_1^2(\theta)) \right\} \quad (95)$$

$$L_5(\theta) = L_3(\theta) \tan \theta \quad (96)$$

$$L_6(\theta) = L_4(\theta) \tan \theta \quad (97)$$

$$\begin{aligned}
L_7(\theta) = \sin \theta \cos \theta \left\{ -\frac{1}{2} \left(\frac{\rho_2^2(\theta)M_1(\theta) - \rho_1^2(\theta)M_2(\theta)}{M_5(\theta)} \right) + \frac{\rho_2(\theta)M_1(\theta) - \rho_1(\theta)M_2(\theta)}{(M_5(\theta))^2} \right. \\
\left. + j \left(\frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^3} \right) + \frac{1}{6} (\rho_2^3(\theta) - \rho_1^3(\theta)) \right\} \quad (98)
\end{aligned}$$

$$\begin{aligned}
L_8(\theta) = \sin \theta \cos \theta \left\{ -\frac{1}{2} \left(\frac{\rho_2^2(\theta)M_1(\theta) - \rho_1^2(\theta)M_2(\theta)}{M_5(\theta)} \right) - j \left(\frac{\rho_2(\theta)M_1(\theta) - \rho_1(\theta)M_2(\theta)}{(M_5(\theta))^2} \right) \right. \\
\left. + \frac{M_1(\theta) - M_2(\theta)}{(M_5(\theta))^3} + \frac{1}{6} (\rho_2^3(\theta) - \rho_1^3(\theta)) \right\} \quad (99)
\end{aligned}$$

for $|\frac{\pi}{a} \cos \theta + k| < \epsilon$.

V. SCATTERING MATRIX - [S]

To determine the scattering parameters S_{ij} for a waveguide fed aperture antenna array radiating into a half-space region, a source is placed in waveguide j and all waveguides are terminated in matched loads. S_{ij} relates the backward wave traveling toward the load in waveguide i to a forward wave traveling toward the aperture in waveguide j . Besides the field directly coupled from waveguide j to i , there are components due to the radiated fields from the other apertures resulting from induced aperture fields. Therefore, the scattering matrix explicitly identifies the incident and reflected (or coupled) wave in every waveguide element.

Let a source which excites only the dominant mode be placed in the j th equivalent waveguide region and represent the fields transverse to the z -direction as

$$\begin{aligned} \underline{E}_{t_j}(z) &= (e^{-\gamma_0 z} - e^{\gamma_0 z}) \underline{e}_0 + \underline{E}_t^{wg}(M_j^j) \\ &= (e^{-\gamma_0 z} - e^{\gamma_0 z}) \underline{e}_0 + V_j \sum_k A_{jk} e^{\gamma_k z} \underline{e}_k \end{aligned} \quad (100)$$

$$\underline{H}_{t_j}(z) = Y_0 (e^{-\gamma_0 z} + e^{\gamma_0 z}) \underline{u}_z \times \underline{e}_0 - V_j \sum_k A_{jk} Y_k e^{\gamma_k z} \underline{u}_z \times \underline{e}_k \quad (101)$$

where the index 0 denotes the dominant mode, γ_k are the modal propagation constants, A_{jk} are the modal constants defined by (25), V_j is the unknown magnetic current coefficient determined by (15), and the Y_k are modal characteristic admittances defined by (34) and (35). The electric field transverse to the z -direction in the i th equivalent waveguide region (no impressed source) is

$$\begin{aligned}
 \vec{E}_{t_i}(z) &= \vec{E}_t^{wg}(M_i^1) \\
 &= V_i \sum_k A_{ik} e^{\gamma_k z} \vec{e}_k .
 \end{aligned} \tag{102}$$

The scattering parameter S_{ij} is the ratio of the backward traveling wave in waveguide i to the forward traveling wave excited by a source in waveguide j . (It is assumed that only the dominant mode wave propagates in the waveguide region.)

For $i = j$, we obtain from (100)

$$S_{ij} = V_j A_{jo} - 1 . \tag{103}$$

For $i \neq j$, we obtain from (100) and (102)

$$S_{ij} = V_i A_{io} . \tag{104}$$

To determine V_i , let us consider equations (15) and (13)

$$[Y^{wg} + Y^{hs}] \vec{V} = \vec{I}^{imp} \tag{15}$$

where

$$\vec{I}^{imp} = [\langle W_j, H_{t_j}^{imp} \rangle]_{N \times 1} . \tag{13}$$

$H_{t_j}^{imp}$ is equal to the first term on the right side of (101) when $z = 0$.

When the aperture is covered by a conductor, the waveguide is terminated by a conducting plane. According to image theory, the tangential magnetic field at $z = 0$ is then just twice the incident wave, or

$$\vec{H}_{t_j}^{imp} = 2 Y_0 \vec{u}_z \times \vec{e}_0 . \tag{105}$$

This is the H_{t_j} used in (13) to evaluate the excitation vector \vec{I}^{imp} .

Hence, the components of the excitation vector are

$$\begin{aligned} I_j^{imp} &= 2 Y_o \iint_{\text{apert.}} \vec{W}_j \cdot \vec{u}_z \times \vec{e}_o \, ds \\ &= 2 Y_o A_{jo} \quad \text{if } i = j \end{aligned} \quad (106)$$

$$I_i^{imp} = 0 \quad \text{if } i \neq j. \quad (107)$$

For an equally spaced linear array, the matrix $[Y^{wg} + Y^{hs}]$ in (15) is symmetric Toeplitz. Equation (15) is solved by a recursion relation given by Zohar [11] which is based on an algorithm by Trench [12].

For a uniformly spaced rectangular array, the matrix $[Y^{wg} + Y^{hs}]$ in (15) is symmetric block-Toeplitz. It is symmetric since we have used a Galerkin solution (choosing expansion functions equal to the testing functions). The block-Toeplitz property for a uniformly spaced rectangular array is given by a matrix of the form

$$[Y] = \begin{bmatrix} Y_o & Y_1 & \dots & Y_n \\ Y_1 & Y_o & Y_1 & \dots & Y_{n-1} \\ \vdots & & & & \\ Y_n & Y_{n-1} & \dots & Y_o \end{bmatrix} \quad (108)$$

where Y_o is that submatrix which defines the self-admittance of an element of the array and $Y_{|i-j|}$, $i \neq j$ is that submatrix which defines the mutual admittance between elements i and j of the array. An element

(submatrix) for the rectangular array is either the shortest row or column of apertures in the x-y plane (see Fig. 5). A recursion relation which uses this block-Toeplitz symmetry property [13] is used to solve (15).

For uniformly spaced isosceles triangular arrays, equation (15) is solved by Gaussian elimination and LU decomposition [14].

It should be mentioned that all three methods used in solving the linear, rectangular, and triangular arrays solve (15) directly without computing the inverse matrix to $[Y^{wg} + Y^{hs}]$.

VI. SAMPLE COMPUTATIONS

A computer program using the formulas derived in the preceding sections has been written. It is described and listed in a subsequent report [15]. In this section we give some examples of the computations that can be made using the general program.

Figure 6 shows the half-space admittance between two narrow slots in echelon $[Y_{12}^{hs}]$ as the separation distance is varied while the angular position is held fixed. This example tested the limiting expressions derived in Section IV since for $\theta = 0$, $(\frac{\pi}{a} \cos \theta - k) = 0$. Our computations are compared to those of Borgiotti [1].

Figure 7 shows the scattering parameter S_{12} (amplitude and phase) for two slots as the angular position of one is varied with respect to the other while the separation distance is held fixed. Our computations are compared to those of Mailloux [3].

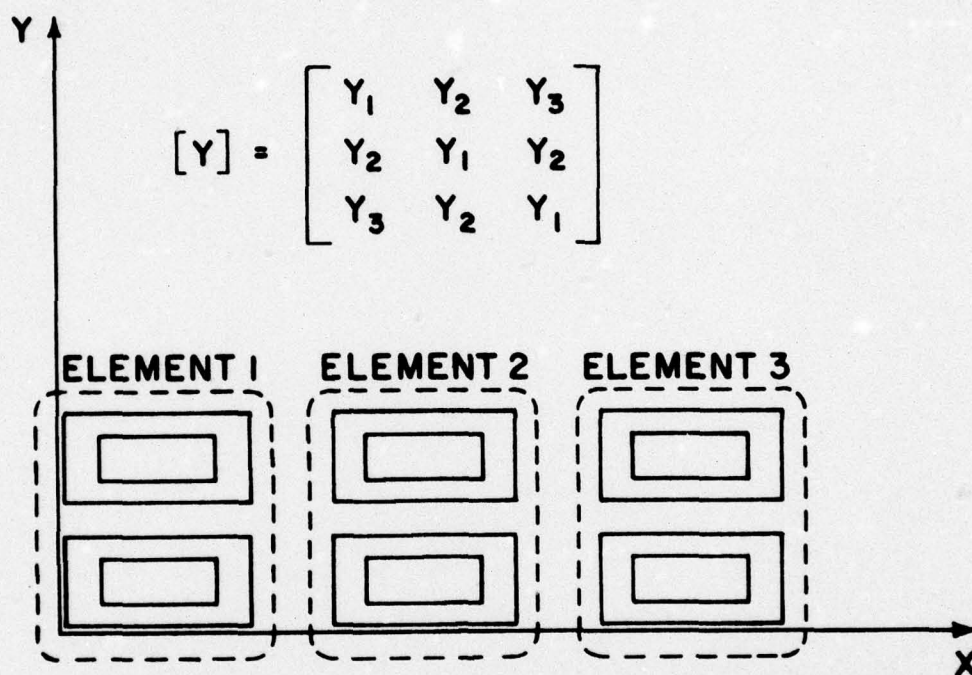


Fig. 5. Choice of elements for a block-Toeplitz admittance matrix for a uniformly spaced 2×3 rectangular aperture array.

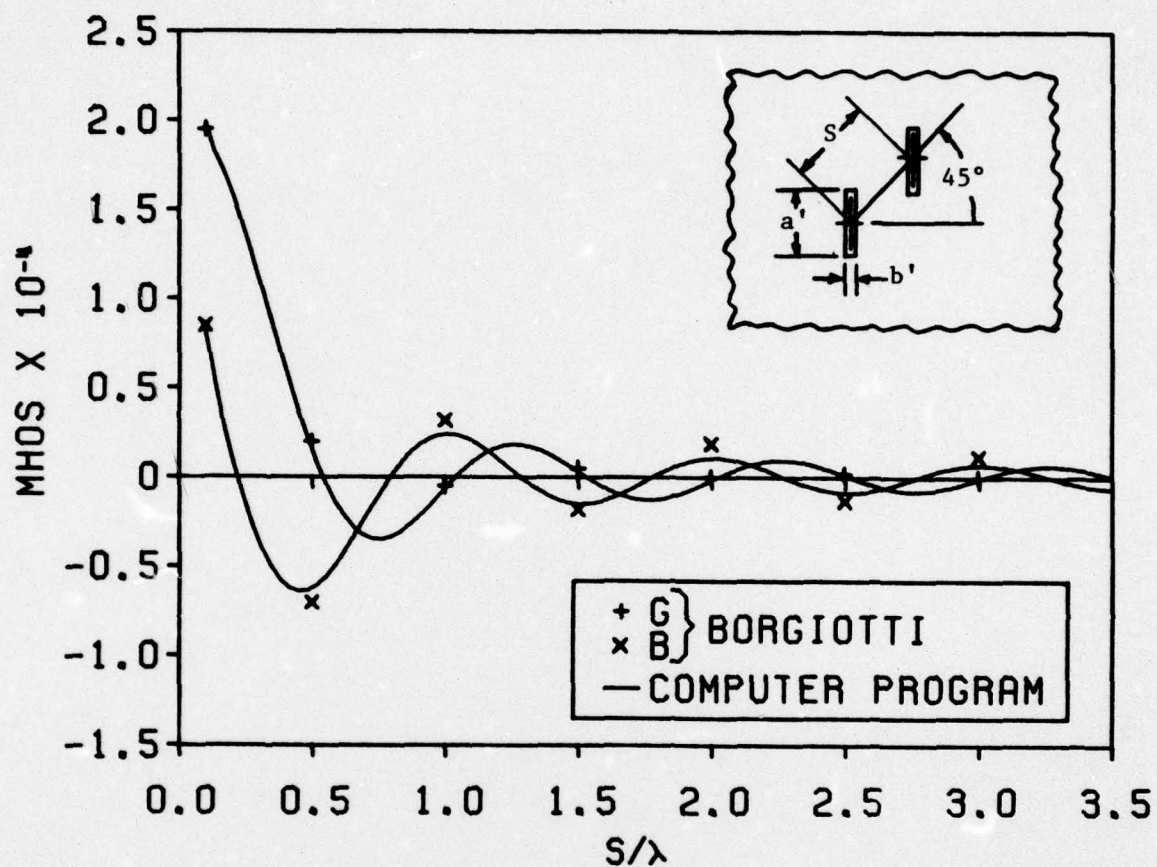


Fig. 6. The half-space admittance between two narrow slots in echelon $[Y_{12}^{hs}]$ where $a'/\lambda = 0.5$ and $b'/\lambda = 0.05$. Our computed results are compared to those calculated by Borgiotti [1].

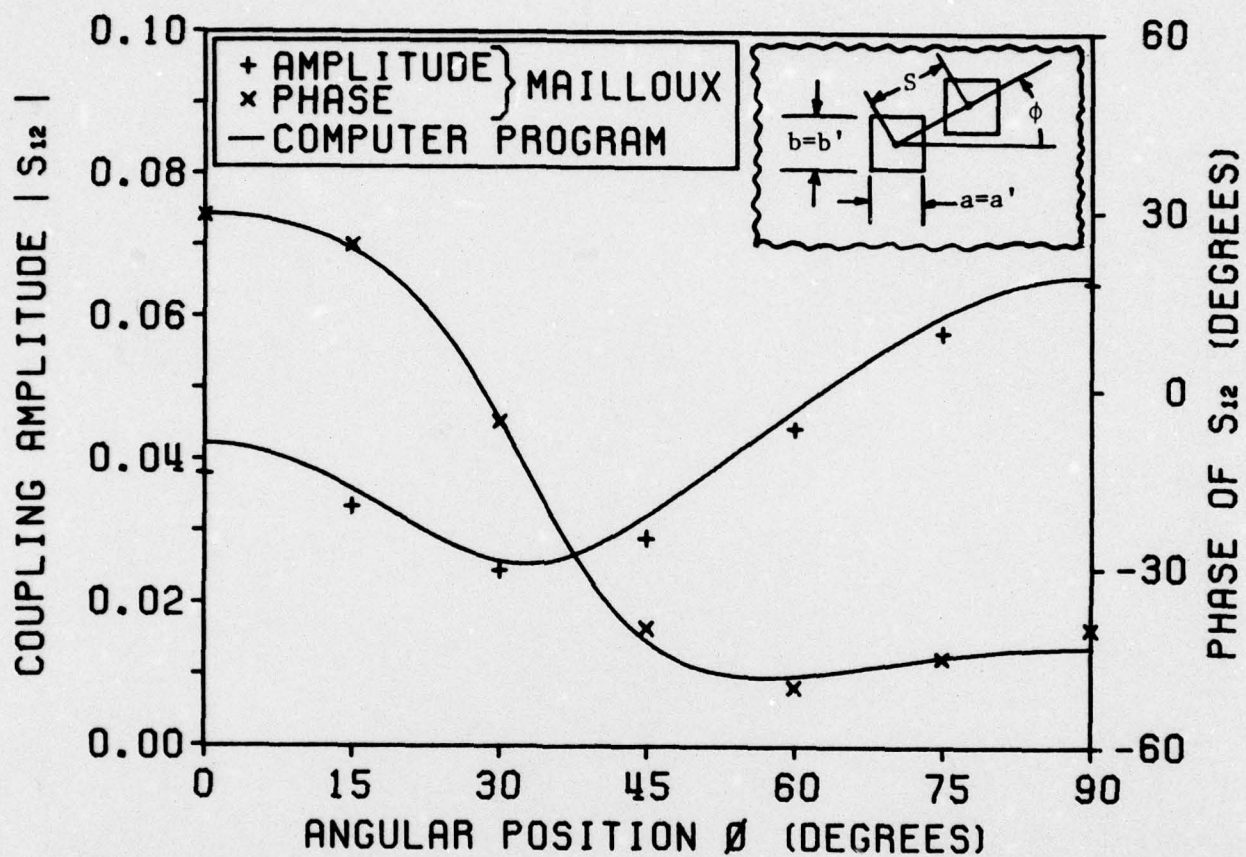


Fig. 7. The scattering parameter S_{12} between two slots where $a/\lambda = a'/\lambda = b/\lambda = b'/\lambda = 0.6$ and $S/\lambda = 0.9$. Our computed results are compared to those calculated by Mailloux [3].

Figure 8 shows the half-space admittance between two slots in echelon $[Y_{12}^{hs}]$ as the separation distance is varied while the angular position is held fixed. Three different slot size cases are shown.

Figure 9 is the same as Figure 8 except that the angular position of one slot with respect to the other is varied while the separation distance is held fixed.

Figure 10 shows the coupled power ($20 \log |S_{i1}|$) and phase of S_{i1} ($i = 2, 3, \dots, 49$) between a driven element and the other elements of a 7×7 waveguide-fed aperture array with a rectangular lattice.

Figure 11 is the same as Figure 10 except that the lattice is now isosceles triangular.

VII. DISCUSSION AND CONCLUSIONS

A method of computing the mutual coupling for an array of uniformly spaced rectangular waveguide-fed apertures radiating into a half-space region has been developed. The array may have a rectangular lattice or an isosceles triangular lattice. The formulation uses one expansion function per aperture so as to be efficient when relatively large numbers of apertures are concerned.

For calculating the half-space self admittance for a single aperture an eight-point Gaussian quadrature numerical integration [16] is performed on the single integrals in (68). The same method of solution is used for calculating the half-space mutual admittance for apertures which are close together (centers of any two given apertures are separated by less than $4a'$). For greater aperture separations a six

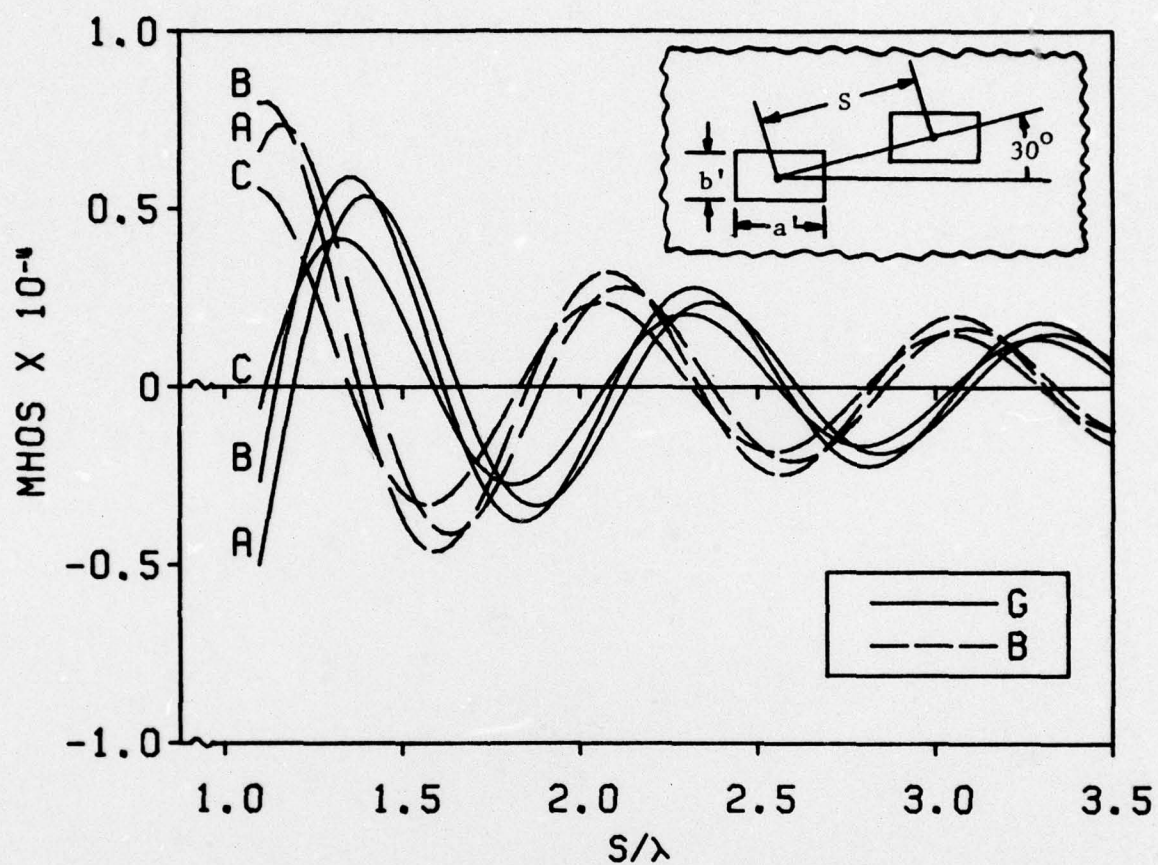


Fig. 8. The half-space admittance between two slots in echelon $[Y_{12}^{hs}]$ where $A = (a'/\lambda = 1.0000, b'/\lambda = 0.4761)$, $B = (a'/\lambda = 0.7500, b'/\lambda = 0.3571)$, and $C = (a'/\lambda = 0.5000, b'/\lambda = 0.2381)$.

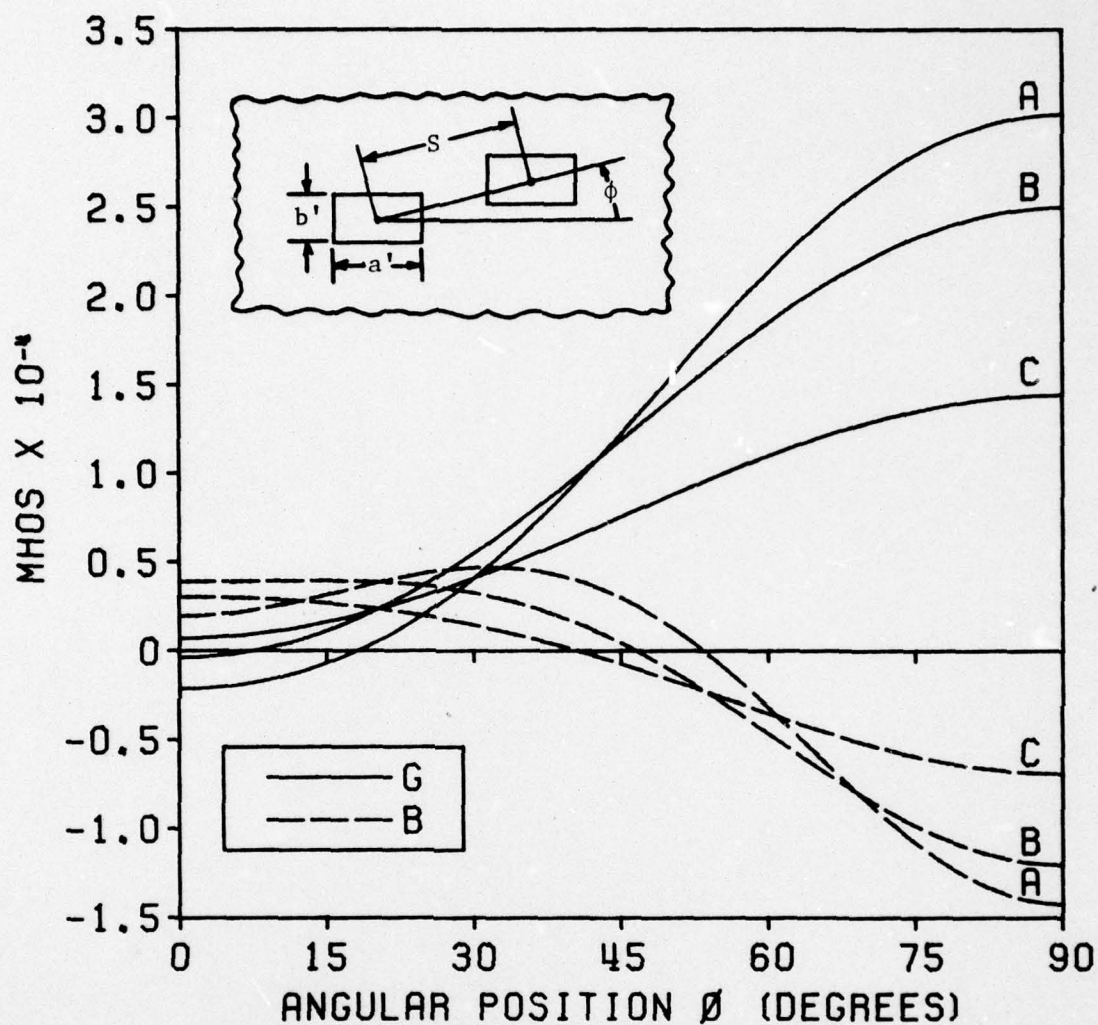


Fig. 9. The half-space admittance between two slots in echelon $[Y_{12}^{hs}]$ where $S/\lambda \approx 1.3$, $A = (a'/\lambda = 1.00, b'/\lambda = 0.4761)$, $B = (a'/\lambda = 0.7500, b'/\lambda = 0.3571)$, and $C = (a'/\lambda = 0.5000, b'/\lambda = 0.2381)$.

-32.6 37	-37.0 -15	-39.9 -160	-43.8 -40	-48.5 31	-52.2 69	-55.7 88
-31.9 -115	-36.6 -171	-40.8 10	-45.0 102	-49.5 147	-53.3 168	-56.9 175
-30.0 98	-35.1 32	-41.9 167	-46.5 -130	-50.9 -103	-55.2 -96	-58.7 -102
-27.1 -49	-33.7 -133	-42.5 -54	-47.5 -15	-53.1 -5	-56.9 -14	-59.9 -23
-23.1 165	-34.2 51	-43.6 75	-50.6 100	-56.3 82	-59.1 77	-64.3 66
-16.6 23	-42.0 -115	-46.9 -158	-54.6 -172	-54.6 170	-61.4 160	-65.6 109
DRIVEN ELEMENT	-36.2 -70	-42.8 -83	-49.8 -116	-63.5 -112	-62.2 137	-65.2 122

Fig. 10. The coupled power ($20 \log |S_{i1}|$) and phase of S_{i1} ($i=2,3,\dots,49$) for a 49 element waveguide-fed aperture array with a rectangular lattice where $a/\lambda = 1.0000$, $a'/\lambda = 0.7500$, $b/\lambda = 0.4761$, $b'/\lambda = 0.3571$, $D_x/\lambda = D_y/\lambda = 0$, and $D_t/\lambda = 0.0515$. The upper number in each aperture represents coupled power in dB while the lower number represents the phase of S_{i1} in degrees.

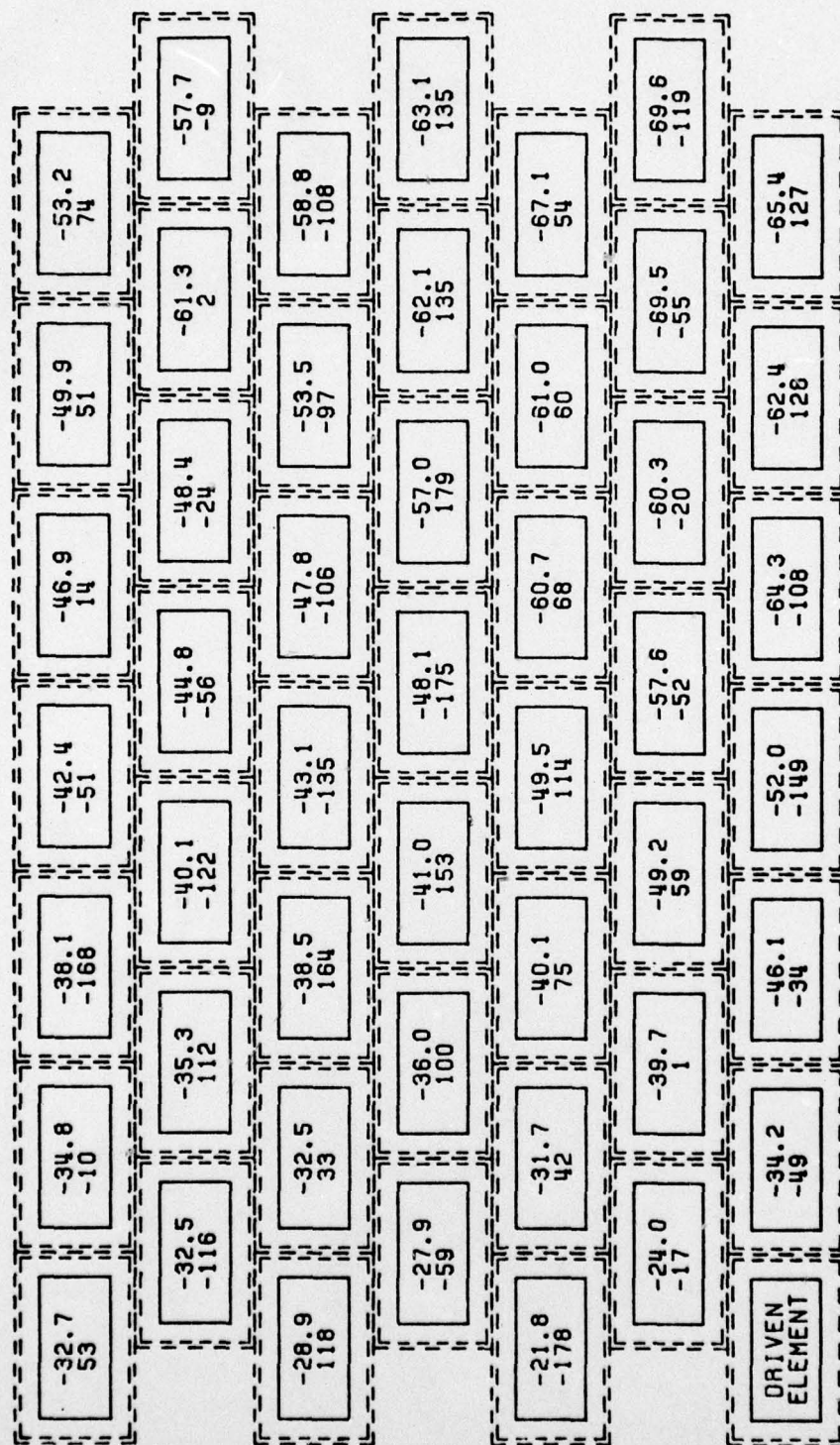


Fig. 11. The coupled power ($20 \log |S_{il}|$) and phase of S_{il} ($i=2,3,\dots,49$) for a 49 element waveguide-fed aperture array with an isosceles triangular lattice where $a/\lambda = 1.000$, $a'/\lambda = 0.7500$, $b/\lambda = 0.4761$, $b'/\lambda = 0.3571$, $D_x/\lambda = D_y/\lambda = 0$, and $D_t/\lambda = 0.0515$. The upper number in each aperture represents the coupled power in dB while the lower number represents the phase of S_{il} in degrees.

point double numerical integration (six points in each variable for a total of 36 points) is performed on the double integrals in (59).

It was found that for apertures which are close together, the solution involving the single integrals of (68) was both more accurate and efficient than the solution involving the double integrals of (59) (an eight point single numerical integration versus a ten point double numerical integration). As the separation distance between apertures increased, the complexity of the solution involving single integrals, specifically, the number of required subdivisions, was not necessary and the solution involving double integrals proved just as accurate and more efficient (a four point single numerical integration versus a six point double numerical integration).

For finding the unknown coefficient vector \vec{V} which is required for determining the scattering matrix $[S]$, the computer program uses Zohar's algorithm [11] for the linear array case and Sinnott's algorithm [13] for the rectangular array case. Both algorithms utilize the Toeplitz symmetry properties of the admittance matrix $[Y^{wg} + Y^{hs}]$. Both algorithms are more efficient than inversion of $[Y^{wg} + Y^{hs}]$ or even a Gaussian elimination-LU decomposition algorithm for large N . (Zohar's algorithm requires approximately $2N^2$ multiplications and divisions, Sinnott's algorithm requires approximately $(n+1)^2 N_p^2$ multiplications and divisions where $n+1$ is the number of blocks of the block-Toeplitz admittance matrix and N_p is the dimension of a block, Gaussian elimination-LU decomposition requires $N^3/3$ multiplications and divisions, and a solution involving inversion of $[Y^{wg} + Y^{hs}]$ requires N^3 multiplications and divisions

for the inverse and another N^2 multiplications for finding \vec{V} given the inverse.) In addition to the efficiency of Zohar's and Sinnott's algorithms, the storage requirements are considerably less than Gaussian elimination-LU decomposition or inversion. (Zohar's algorithm requires the storage of only one row of $[Y^{wg} + Y^{hs}]$ while Sinnott's algorithm requires storing one row of blocks of $[Y^{wg} + Y^{hs}]$.)

If very large arrays are encountered, further simplification of the program could be made by using the present formulation for apertures which are close together, and using a far-field approximation where the field is assumed constant over the aperture for the rest of the apertures.

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